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A multidimensional mereotopological theory

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Abstract

Qualitative calculi are useful for reasoning with spatial data because they are thought to more closely reflect commonsense reasoning than quantitative methods. We propose a calculus that incorporates spatial entities of various dimensions and the relationships between them. Our first-order formalism defines a set of jointly exclusive and pairwise disjoint relations which enable reasoning about space involving objects in a mixture of dimensions.

1 Introduction

Data is increasingly ubiquitous in the modern world. Spatial data is among the most common types of data, and plays a principal role in many interdisciplinary fields such as artificial intelligence, and in a range of application areas, including land use planning and disaster management. Spatial data can be collected either manually or automatically from a range of different sources such as satellite-based remote sensing, cameras and field surveys. Since collection methods use various formats and approaches to store geospatial data, data integration is needed, leading to the introduction of geospatial data standards such as OGC/ISO Simple Features [Open Geospatial Consortium (OGC), 2010] and OGC's GeoSPARQL [Open Geospatial Consortium (OGC), 2012] and GML [Gröger *et al.*, 2012]. Geographic objects are represented as geometries, including simple and complex geometric features, in the above mentioned standards. The most typically employed features are points, curves (1-dimension), surfaces (2-dimensions), solids (3-dimensions) and their aggregations for each dimension. These geometries are implemented and queried in geographic information systems (GIS) and spatial databases to support spatial queries and different forms of analysis. Spatial queries allow the user to ask the system to identify spatial relationships such as distance (length), (mereo)topology, etc. between geometries.

Mereotopological relations (which return Boolean values to indicate whether a relation exists of not) allow humans to access geospatial data in a more intuitive way than purely numerical methods. They have been incorporated into many geographic standards and include relations such as intersects, contains, overlaps, meets or crosses (the latter being only valid between 1D features). These are based on qualitative theories of space and represent equivalent mereotopological relations whether via the 9-intersection model (9IM) [Egenhofer, 1991; Egenhofer and Herring, 1991], its dimensionally extended version (DE-9IM) [Clementini and Di Felice, 1995; Clementini *et al.*, 1993], or RCC [Randell *et al.*, 1992].

City GML [Gröger *et al.*, 2012] is an OGC standard for vector based encoding of georeferenced 3-dimensional city data incorporating relevant semantics. It is an extension of the simple feature access standard [Open Geospatial Consortium (OGC), 2010] (which is limited to two-dimensional geometries) that accepts three-dimensional geometries. It is at least partly applied by a variety of spatial database systems and geographic information systems such as ArcSDE (the spatial database of ArcGIS), PostGIS, MySQL, Oracle and IBMDb2. However, while the mereotopological relations in City GML (and other standards) provide an answer to a spatial query in three dimensions, they do not support qualitative reasoning, as there is no formal link between the mereotopological and geometrical representations. Moreover, this deficiency causes loss of meaning in the data exchange process. For instance, information collected from two sources, "Road A is flooded and passes Adams Park" and "Road B that borders Adams Park is not flooded" can not be analyzed in GIS without the relationship between the geometries and named features being understood.

The traditional qualitative spatial representation and reasoning frameworks, such as the RCC, support reasoning about topological relations between features of heterogeneous dimensions, but do not support reasoning among features with different dimensions. While logical theories of multidimensional mereotopologies that are capable of modeling mereotopological relations between geometric spatial features of any dimensionality (e.g., polygons, solids) have been proposed by Galton [1996, 2004]; Gotts *et al.* [1996]; Hahmann and Grüninger [2011]; Hahmann [2018], no qualitative spatial *calculi* for composition-based reasoning with features of multiple dimensions is yet available. We propose a multidimensional mereotopological theory of space that yields a lattice of relations for composition-based qualitative spatial reasoning. Our contributions are (1) a first-order logic ontology inspired by City GML features' definitions and (2) a lattice of

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mereotopological relations that are derived using automated theorem provers (ATPs) and that support composition-based qualitative reasoning.

This paper is structured as follows. In Section 2, we discuss previous relevant work. In Section 3, we briefly present some preliminaries to our theory. Section 4 introduces our formalism. We check the consistency of the theory in section 5. Finally, Section 6 presents possible opportunities for extending this research.

2 Background and Related Work

Mereotopological relations form a fundamental aspect of qualitative spatial reasoning [Harmelen *et al.*, 2008]. They incorporate mereological relations like parthood, inside, or containment and topological relations such as connection/contact and disconnection, along with bridging relations (e.g., partially overlap) between these two aspects.

Initial efforts [Egenhofer, 1991; Egenhofer and Herring, 1991; Clementini and Di Felice, 1995; Clementini *et al.*, 1993, 1994] to model topological relations between spatial features have been implemented in geospatial data standards. They follow an intersection strategy over features in heterogeneous dimensions. For example, 9IM [Egenhofer, 1991; Egenhofer and Herring, 1991] compares the existence (emptiness or non-emptiness) of the intersection between the "interior", "boundary" and "exterior" parts of two features. The dimensional value of these common parts is used in DE-9IM [Clementini and Di Felice, 1995; Clementini *et al.*, 1994] (i.e., emptiness: -1D, Point: 0D, Line: 1D and Surface: 2D). However, intersections between the features' exteriors are removed in CBM [Clementini *et al.*, 1993] and replaced by a description of the shared part in terms of the participant individuals (i.e., whether equal to one of them). This way of identifying topological relations results in a large number of relations (e.g., 512 relations in 9IM), because of the consideration of every combination of the intersections. Large numbers of relations like this are difficult for users to work with, can contain a lot of redundancy and is computationally inefficient.

Moreover, this approach requires some operations over all the features to extract their interior and boundary parts. Furthermore, the associated semantics of the introduced mereotopological relations (e.g., crosses is specialized by overlap) have not been clarified. The lack of representing relevancy among the relations causes a problem in qualitative reasoning.

In contrast, qualitative relations and their semantic interrelations can also be described by axiomatic systems [Eschenbach, 2001]. These systems are based on a limited number of primitive relations (e.g., connection or parthood) with their associated interpretations. Further relations such as overlap are then defined based on the primitive relation or relations. Explicit formalization of the interrelations among the mereotopological relations in the axiomatic frameworks allows qualitative reasoning without any geometric information. RCC [Randell *et al.*, 1992] is a well-known axiomatic system in the qualitative spatial domain. It introduces a set of eight topological relations between equi-dimensional features, which are the same as the ones introduced by 9IM between two-dimensional entities. RCC (as well as other spatial axiomatic theories) follows human cognition in representing space, in which spatial entities intuitively represent objects. These entities are typically known as regions, and no dimensional difference between them is permitted in the traditional axiomatic frameworks (i.e., only relations between objects of the same dimensions are supported).

Multidimensional axiomatic theories aim to overcome the above mentioned shortcomings by proposing a set of mereotopological relations between entities of different dimensions [Gotts, 1996; Galton, 1996; Smith, 1996; Hahmann and Grüninger, 2011; Hahmann, 2018]. The INCH calculus [Gotts, 1996] is a first-order theory based on a single primitive, INCH, interpreted as "includes a chunk of". Its idea has been more fully developed in CODIB [Hahmann, 2013] by employing a more natural set of primitives (i.e., containment, relative dimension and boundary containment) to represent entities of different dimensions and the relationships between them, including their boundaries. However, CODIB does only specify a set of four contact relations that are jointly exhaustive and pairwise disjoint (JEPD) but does not specify a complete set of mereotopological relations (including parthood relations) that are JEPD, that is, such that each pair of entities in a spatial scene is related by one and only one relation. A complete set of JEPD relations are essential to compute a compositional table and perform composition-based qualitative reasoning. Moreover, the defined algebraic structure based on JEPD relations makes use of composition and converse operations for constraint propagation. Other attempts (namely Smith [1996]; Galton [1996]) offer higher-order formalization of the space, which are more expressive than their counterparts (first-order theories). However, they are not decidable and thus can not be used for qualitative reasoning. Furthermore, the existence of lower-dimensional entities depends on higher-dimensional ones in these theories, i.e., a surface can not be available without referring it to a (boundary) of a solid. The proposed set of relations in our first-order formalism, unlike other multidimensional theories, has the JEPD property, so the theory can handle qualitative reasoning, specifically offering a composition table that allows the result of combinations of spatial relations to be looked up.

3 Preliminaries

We now describe our notion of space and its contents. Following Newton's view, we adopt an absolute model of space in which its existence is independent of its content. Indeed, it is an individual entity in its own right, which could exist even if empty, with no inhabitants [Casati and Varzi, 1999].

Elements of the space are 'regions' that are extended in zero, one, two and three dimensions. We name them a point, line, polygon and polyhedral, respectively. They are not necessarily disjoint sets of regions (i.e., they may overlap each other and occupy the same space, but they could be disjoint). Spatial regions express mathematical parts of space that are apart from an actual physical objects that occupy the space. Moreover, regions are 'regular' which means that their parts are in the unify dimension, e.g., a two-dimensional extended entity with a tail of one-dimension does not consider as a region of the space. Regions (except points) are considered to be 'closed'. By 'closed' we mean that the limits (boundary) of a region are elements of it. Furthermore, the space itself is a special region having the highest extension (i.e., three-dimension), and not having any boundary.

Besides, only 'simple' regions are in the scope of this work. They are both internally and externally connected, i.e., not having any holes and only consisting of a single piece. Collections of simple regions of any dimension are out of the scope.

The labels of the presented logical sentences are formatted as '[type] [theory section][number]' (e.g., D IP1) where type can be axiom (A: accepted statement), definition (D: giving a meaning to a relation or concept) or theorem (T: a provable property from the axioms and definitions), while the second group of letters indicate the section of theory (e.g., P = parthood, BP = boundary part, IP = interior part). All sections of the theory are being investigated in the context of the COLORE (Common Logic Ontology Repository).

4 Formalism

In this Section, we present the core of our formalism in first-order logic. The mereotopological relations are dimensionally independent here. Moreover, our theory is developed based on two primitives: the parthood relation (**P**) which is common in traditional mereotopological theories, and the boundary part relation (**B**) which is added here to represent the extremity of the (non-zero) extended regions. The following subsections will describe these primitives, their supporting axioms and theorems and a set of defined relations based on them.

4.1 Parthood

The first primitive relation is *parthood*, '$\mathbf{P}xy$', which reads as '*x* is part of *y*'. The parthood relation holds even when *x* and *y* are identical. It is a reflexive, anti-symmetric and transitive relation – a partial order.

A P1. $\forall x\ (\boldsymbol{P}xx)$

A P2. $\forall x \forall y\ (\boldsymbol{P}xy \wedge \boldsymbol{P}yx \rightarrow x = y)$

A P3. $\forall x \forall y \forall z\ (\boldsymbol{P}xy \wedge \boldsymbol{P}yz \rightarrow \boldsymbol{P}xz)$

So, as an immediate consequence of the anti-symmetric property of **P**, we have:

T P1. $\forall x \forall y\ (x = y \leftrightarrow \boldsymbol{P}xy \wedge \boldsymbol{P}yx)$

By the introduction of the identity (**T P1**), the formalism can be presented in first-order logic without equality in light of the explicit definition of the identity relation as:

D P1. $\forall x \forall y\ (x = y \equiv_{def} (\forall z\ \boldsymbol{P}zx \leftrightarrow \boldsymbol{P}zy))$

Also, a number of further mereological relations can be defined based on the 'parthood' primitive:

D P2. *x* ***overlaps*** *y:* $\forall x \forall y\ (\boldsymbol{O}xy \equiv_{def} (\exists z\ (\boldsymbol{P}zx \wedge \boldsymbol{P}zy)))$

D P3. *x is* ***discrete*** *from y:* $\forall x \forall y\ (\boldsymbol{D}xy \equiv_{def} \neg \boldsymbol{O}xy)$

D P4. *x is a* ***proper part of*** *y:* $\forall x \forall y\ (\boldsymbol{PP}xy \equiv_{def} \boldsymbol{P}xy \wedge \neg \boldsymbol{P}yx)$

D P5. *y* ***has a part*** *x:* $\forall x \forall y\ (\boldsymbol{P}^{-1}xy \equiv_{def} \boldsymbol{P}yx)$

D P6. *y* ***has proper part*** *x:* $\forall x \forall y\ (\boldsymbol{PP}^{-1}xy \equiv_{def} \boldsymbol{PP}yx)$

D P7. *x is a* ***point****:* $\forall x\ (\boldsymbol{Pt}x \equiv_{def} \forall y\ (\boldsymbol{P}yx \rightarrow y = x)$

The overlap relation holds when there is at least one common region between the participants. It is reflexive (from reflexivity of **P** and **D P2**) and symmetric (from **D P2**) but not transitive. The discrete relation represents a case in which there is no shared region between the participants. It is proved to be irreflexive (from reflexivity of **P** and **D P3**) and symmetric (from symmetry of **O** and **D P3**).

The proper part relation excludes the identity case from the parthood relation (**P**). Contrary to the overlap relation, the proper part relation is transitive but irreflexive and asymmetric-a strict partial order. These properties are easily verified by **D P4** and properties of **P**. The non-symmetric properties of **P** and **PP** result in the introduction of their converse $\mathbf{P}^{-1}$ and $\mathbf{PP}^{-1}$, respectively.

Adding point (**D P7**) to the given mereological theory is not only necessary (both for representing the zero-dimension extended regions and for comparison to other multidimensional theories), but also clarifies the chosen interpretations for the mereotopological relations, for instance, allowing the different types of overlap relation to be valid for two connected one-dimensional regions. According to this definition and **D P4**, a point does not have any other proper part than itself.

Our theory is further restricted by an axiom saying that if a region has a proper part, that proper part region must be supplemented by another discrete proper part (supplementation principle):

A P4. $\forall x \forall y\ (\boldsymbol{PP}xy \rightarrow \exists z\ (\boldsymbol{P}zy \wedge \boldsymbol{D}zx))$

As a result of the supplementation principle, we can say that the regions with the same parts are identical (*extensionality theorem*):

T P2. $\forall x \forall y\ (\exists z\ (\boldsymbol{PP}zx \wedge \boldsymbol{PP}zy)) \rightarrow (\forall z\ (\boldsymbol{P}zx \leftrightarrow \boldsymbol{P}zy) \rightarrow x = y)$

Also, the following theorem is immediately provable from the definition of the overlap relation (**D P2**) showing that parthood is a special case of overlap:

T P3. $\forall x \forall y\ (\boldsymbol{P}xy \rightarrow \boldsymbol{O}xy)$

To make the algebraic structure of the theory neater on one hand and make the defined operations[1] total in the domain, we define '***null***' (a lower bound of the domain) as an individual not being part of anything:

D P8. ***null*** $=_{def} \iota x(\forall y\ (\neg \boldsymbol{P}xy \wedge \neg \boldsymbol{P}yx))$

The theory also includes an axiom that as its result the domain of spatial entities has an upper bound. It is an individual (*universal* region) that has everything as its part (except the ***null*** entity):

A P5. $\exists z \forall x\ (x \neq \textbf{\textit{null}} \rightarrow \boldsymbol{P}xz)$

From the axioms for identity[2] and in the presence of **T P2**, the *universal* region is unique and defined as:

D P9. $\boldsymbol{U} =_{def} \iota x(\forall y\ (\boldsymbol{P}yx))$

[1] We will define Boolean operators and some other operators latter on.

[2] The standard identity relation (=) is reflexive, symmetric and transitive.

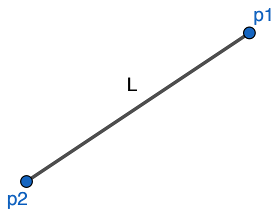

Figure 1: Boundary of a line segment.

4.2 Boundary Part

The second primitive is *boundary part of*, '$\mathbf{B}xy$', that reads as 'x is a boundary part of y'. We say 'boundary part of' and not 'boundary of' to permit boundaries that are not *maximal* (corner points, edges, surfaces)[3]. $\mathbf{B}xy$ is stipulated as:

A BP1. $\forall x \forall y\ (\boldsymbol{B}xy \rightarrow \boldsymbol{PP}xy)$

A BP2. $\forall x \forall y \forall z\ (\boldsymbol{B}xy \wedge \boldsymbol{B}yz \rightarrow \boldsymbol{B}xz)$

A BP3. $\forall x \forall y \forall z\ (\boldsymbol{P}xy \wedge \boldsymbol{B}yz \rightarrow \boldsymbol{B}xz)$

A BP4. $\forall x \forall y \forall z\ (\boldsymbol{B}xy \wedge \boldsymbol{P}yz \rightarrow \boldsymbol{P}xz)$

A BP1 says that the boundary part of a region is a proper part of it, which is compatible with the assumption of the closed regions. For instance, if y is a simple line segment (without any self-intersection), then $\mathbf{B}xy$ represents its endpoints in x, e.g., points p_1 and p_2 are the boundary of the line segment L in Figure1). Transitivity of the boundary parts is shown in **A BP2**, which says that any region bounding a region that bounds z also bounds z. **A BP3** says that any part of a region bounding z, is also a boundary of z, on the other hand, **A BP4** says that any boundary part of a region which itself is parts of z, is still parts of z. Intuitively, all regions that are smaller than the universe (**D P8**) must have a boundary part (except points and ***null***).[4]

A BP5. $\forall y\ (\neg \boldsymbol{Pt}y \wedge y \neq \boldsymbol{U} \wedge y \neq \boldsymbol{null} \rightarrow \exists x\ (\boldsymbol{B}xy))$

It can be said trivially (from **A BP1** and **D P4**) that a boundary of a region is its part:

T BP1. $\forall x \forall y\ (\boldsymbol{B}xy \rightarrow \boldsymbol{P}xy)$

Intuitively, we consider the universal region (defined in **D P8**) and point (defined in **D P7**) to be unbounded in (**A BP6**) and (**A BP7**) respectively:

A BP6. $\neg \exists x\ (\boldsymbol{B}x\boldsymbol{U})$

A BP7. $\forall x\ (\boldsymbol{Pt}x \rightarrow \neg \exists y\ (\boldsymbol{B}yx))$

A BP5 also says that every non-universal region (and non-null entity) is either a point or has a boundary part:

T BP2. $\forall x\ (x \neq \boldsymbol{U} \wedge x \neq \boldsymbol{null} \rightarrow \boldsymbol{Pt}x \vee (\exists z\ (\boldsymbol{B}zx))$

Similarly, the non-universal region (and non-null entity) who have boundary are not points:

T BP3. $\forall x \forall y\ (x \neq \boldsymbol{U} \wedge x \neq \boldsymbol{null} \wedge \boldsymbol{B}yx \rightarrow \neg \boldsymbol{Pt}x)$

[3] We will define the maximal boundaries via **Bd**(x) later.

[4] y could not be a collection of points in this axiom since it is not part of our domain. We intend to address collections of equidimensional regions in future work.

The proper part relation can be specialized by using the boundary part relation. It represents the spatial configuration in which a region connects another one from the inside, *tangential proper part* relation (**TPP**).

D BP1. $\forall x \forall y\ (\boldsymbol{TPP}xy \equiv_{def} \boldsymbol{PP}xy \wedge (\exists z(\boldsymbol{P}zx \wedge \boldsymbol{B}zy)))$

Because of the non-symmetric property of **TPP**, its converse is:

D BP2. $\forall x \forall y\ (\boldsymbol{TPP}^{-1}xy \equiv_{def} \boldsymbol{TPP}yx)$

Note that all boundary parts are tangential proper parts, but not vice versa.

T BP4. $\forall x \forall y\ (\boldsymbol{B}xy \rightarrow \boldsymbol{TPP}xy)$

4.3 Interior Part

According to Brentano's suggestion [Brentano, 1988], boundaries are dependent regions and do not exist without the region they bound. In conformity with his thesis, for every boundary, there is an entity which it bounds and that entity has an interior part. So, we need to define an *interior* (**I**) before capturing the idea of dependency:

D IP1. $\forall x \forall y\ (\boldsymbol{I}xy \equiv_{def} \boldsymbol{PP}xy \wedge \neg \boldsymbol{B}xy \wedge \forall z\ (\boldsymbol{B}zy \rightarrow \boldsymbol{D}zx))$

In other words, an interior part of a region is its part but not its boundary part, and it has no common part with its boundary part. Immediately, we can say that if some region is an interior part of another region, it is a part of it.

T IP1. $\forall x \forall y\ (\boldsymbol{I}xy \rightarrow \boldsymbol{P}xy)$

The **D IP1** also yields that interior and boundary parts of a region are disjoint:

T IP2. $\forall x \forall y\ (\boldsymbol{I}xy \rightarrow \neg \boldsymbol{B}xy)$

T IP3. $\forall x \forall y\ (\boldsymbol{B}xy \rightarrow \neg \boldsymbol{I}xy)$

Intuitively, we are expecting that every proper part of a region is either its interior part or tangential proper part (from reflexivity of **P**, **D P2**, symmetric of **O**, **D P3**, **D P4**, **D BP1** and **D IP1**):

T IP4. $\forall x \forall y\ (\boldsymbol{PP}xy \rightarrow \boldsymbol{I}xy \vee \boldsymbol{TPP}xy)$

Moreover, we can say that the universal region (**U**) is an interior part of itself:

A IP1. ***IUU***

Consequently, all the other regions are interior parts of the universal region (from irreflexively of **PP**, **D IP1** and **A IP1**):

T IP5. $\forall x\ (\boldsymbol{I}x\boldsymbol{U})$

All parts of a region y which is entirely in the interior part of a region z are also the interior parts of z (from properties of **P**, **D P2**, **D P3**, **D P4**, **D IP1** and **T IP5**):

T IP6. $\forall x \forall y \forall z\ (\boldsymbol{P}xy \wedge \boldsymbol{I}yz \rightarrow \boldsymbol{I}xz)$

Similarly, all interior parts of a region y which is completely contained in a region z are also part of z (from transitivity of **P** and **D P4**):

T IP7. $\forall x \forall y \forall z\ (\boldsymbol{I}xy \wedge \boldsymbol{P}yz \rightarrow \boldsymbol{P}xz)$

There is a converse relation ($\mathbf{I}^{-1}xy$) for the **I** says that x has y as an interior part.

D IP2. $\forall x \forall y\ (\boldsymbol{I}^{-1}xy \equiv_{def} \boldsymbol{I}yx)$

Up to now, we have talked about parts of a region that are either interior part or bounding part of it, but we have not mentioned *the boundary of* (i.e., the whole boundary of) a region. Indeed, it is a region which represents *all* the limits of the bounded region and so it is *maximal*. It is defined via a *boundary* operation (bdy) purely in terms of the **B** relation:

D IP3. $\mathtt{bdy}(x) =_{def} \iota y\ (y = \boldsymbol{null} \vee (\boldsymbol{B}yx \wedge (\forall z\ (\boldsymbol{B}zx \rightarrow \boldsymbol{P}zy))))$

in which $y = \boldsymbol{null}$ accounts for the special case where $\mathtt{bdy}(x)$ is empty. The boundary is empty for points, the universal region and ***null*** entity. Also, the enclosed regions such as a ring (i.e., 1D region) have empty boundary.

A IP2. $\forall x\ (\mathtt{bdy}(x) = \boldsymbol{null} \leftrightarrow x = \boldsymbol{null} \vee x = \boldsymbol{U} \vee \boldsymbol{Pt}x \vee (\forall z\ (\boldsymbol{PP}zx \rightarrow \boldsymbol{I}zx))$

The uniqueness and maximality of this operator are guaranteed by the following axioms respectively:

A IP3. $\forall x \forall y \forall z\ (\mathtt{bdy}(x) = y \wedge \mathtt{bdy}(x) = z \rightarrow z = y)$

and:

A IP4. $\forall x \forall w\ (\boldsymbol{O}w\mathtt{bdy}(x) \leftrightarrow \exists v\ (\boldsymbol{O}vw \wedge \boldsymbol{B}vx))$

According to **D IP3**, every contributory part of y is part of the maximal boundary. So, vertices of a polygon are also elements of a closed curve, which is the (maximal) boundary for the polygon. Now, we can express the dependency thesis to the effect that each boundary is such that we can identify an entity being isolated by, and that has interior components. However, we first define the predicate "is a boundary"as a unary relation that identifies whether something is the boundary of anything at all:

D IP4. $\forall x\ (\boldsymbol{Bd}x \equiv_{def} \exists y\ (\boldsymbol{B}xy))$

Then we can state the dependency idea as (from **D IP3** and **D IP4**):

T IP7. $\forall x\ (\boldsymbol{Bd}x \rightarrow \exists z \exists t\ (\boldsymbol{B}xz \wedge \boldsymbol{I}tz))$

Since our theory is intended to describe the spatial relations in a multidimensional space, we have various kinds of relations that are representing different types of overlap in the spatial configuration: *imbricate* (**IMB**) covers all the overlap configurations excluding part of and its converse, *externally connected* (**EC**) describes the cases in which two spatial regions *only* meet in their boundary parts; and *partially overlap* (**PO**) represents the arrangements where the interior-interior intersection is also possible as well as boundary-boundary intersection.

D IP5. $\forall x \forall y\ (\boldsymbol{IMB}xy \equiv_{def} \boldsymbol{O}xy \wedge \neg \boldsymbol{P}xy \wedge \neg \boldsymbol{P}yx)$

D IP6. $\forall x \forall y\ (\boldsymbol{EC}xy \equiv_{def} \boldsymbol{IMB}xy \wedge (\forall z\ (\boldsymbol{P}zx \wedge \boldsymbol{P}zy \rightarrow (\boldsymbol{B}zx \wedge \boldsymbol{B}zy)))$

D IP7. $\forall x \forall y\ (\boldsymbol{PO}xy \equiv_{def} \boldsymbol{IMB}xy \wedge (\exists p\ (\boldsymbol{I}px \wedge \boldsymbol{I}qy) \wedge (\exists q\ (\boldsymbol{P}qx \wedge \boldsymbol{B}qy)) \wedge (\exists t\ (\boldsymbol{B}tx \wedge \boldsymbol{P}ty)))$

The imbricate relation can be refined further to represent the cases in which there is *only* interior-interior intersection between two objects, which is referred to as the *crosses* relation (**X**), and the configurations in which there is a boundary-interior intersection between them referred to as the *overpass* relation (**OV**).

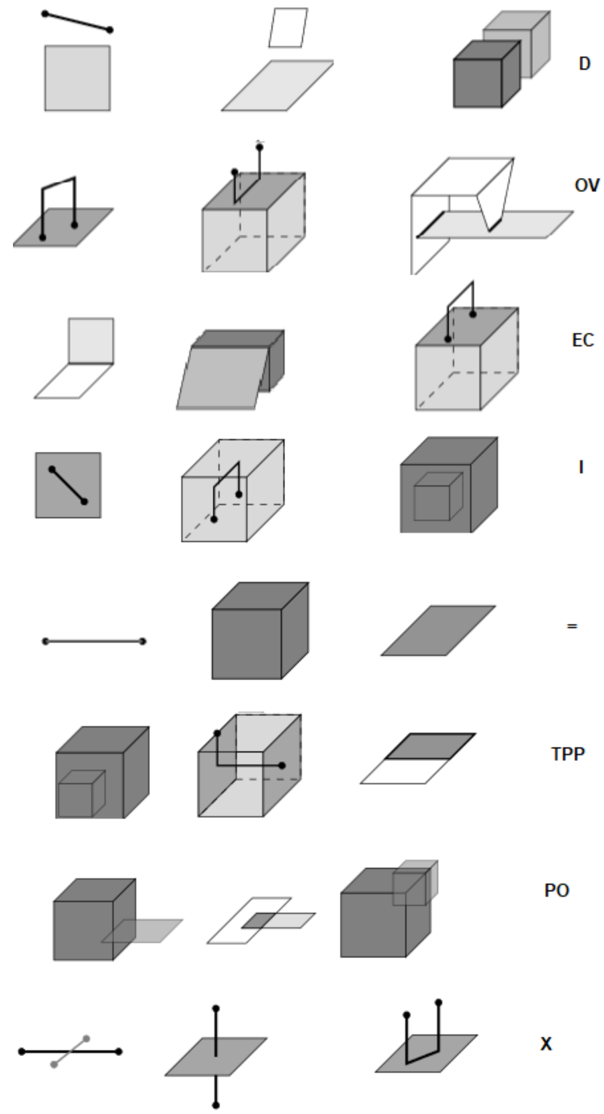

Figure 2: Illustration of some of the relations between regions *a* and *b*.

D IP8. $\forall x \forall y\ (\boldsymbol{X}xy \equiv_{def} \boldsymbol{IMB}xy \wedge (\forall z\ (\boldsymbol{P}zx \wedge \boldsymbol{P}zy \rightarrow (\boldsymbol{I}zx \wedge \boldsymbol{I}zy))))$

D IP9. $\forall x \forall y\ (\boldsymbol{OV}xy \equiv_{def} \boldsymbol{IMB}xy \wedge \neg(\exists p\ (\boldsymbol{I}px \wedge \boldsymbol{I}qy)) \wedge (\exists s\ (\boldsymbol{B}sx \wedge \boldsymbol{I}sy) \vee \exists s\ (\boldsymbol{I}sx \wedge \boldsymbol{B}sy)))$

Examples of the proposed relations of our theory are shown in Figure 2.

According to the axioms and theorems mentioned above, we can define Boolean operations[5] as below. Introduction of the ***null*** makes all of these operations total (i.e., definable in the domain): $\mathtt{sum}(x, y)$ is read as "the sum x and y", $\mathtt{prod}(x, y)$ is read as "the product(intersection) of x and y, $\mathtt{compl}(x)$ is read as "the complement of x" and $\mathtt{diff}(x, y)$

[5]All the definitions using ι are translated into object language to keep the formalism in FOL. For example, the sum is translated into $\forall x \forall y\ (\forall w\ \mathbf{O}(w, \mathtt{sum}(\mathrm{x,y})) \leftrightarrow \mathbf{O}(w, x) \vee \mathbf{O}(w, y)))$

is read as "the difference of x and y".

D IP11. $\forall x \forall y\ (\mathtt{sum}(x, y) =_{def} \iota z(\forall w\ (\boldsymbol{P}wz \leftrightarrow \boldsymbol{P}wx \vee \boldsymbol{P}wy)))$

D IP12. $\forall x \forall y\ (\mathtt{prod}(x, y) =_{def} \iota z(\forall w\ (\boldsymbol{P}wz \leftrightarrow \boldsymbol{P}wx \wedge \boldsymbol{P}wy)))$

D IP13. $\forall x\ (\mathtt{compl}(x) =_{def} \iota z(\forall w\ (\boldsymbol{P}wz \leftrightarrow \boldsymbol{D}wx \wedge \boldsymbol{P}wz)))$

D IP14. $\forall x \forall y\ (\mathtt{diff}(x, y) =_{def} \iota z(\forall w\ (\boldsymbol{P}wz \leftrightarrow \boldsymbol{P}wx \wedge \boldsymbol{D}wy)))$

4.4 Relational Lattice

The existence of upper and lower bounds for the domain with a *poset* relation (**P**) enables us to derive a lattice of relations from the axiomatic theory. The lattice depicts the links among the defined mereotopological relations according to two rules: *specialization* (rule I) and *subsumption* (rule II).

I) Where there is an edge between two relations in a lattice, some source relation **S** (lower in the lattice) implies the target relation **T** (further up in the lattice):

$$\mathbf{S}xy \rightarrow \mathbf{T}xy$$

For example, $\mathbf{PO}xy \rightarrow \mathbf{O}xy$.
When one relation points to (i.e., specializes) more than a single relation (e.g., $=$ specializes **P** and $\mathbf{P}^{-1}$), then the specialized relation implies all of the relations it points to. For example, $x\mathbf{=}y \rightarrow \mathbf{P}xy \wedge \mathbf{P}^{-1}xy$.

II) Where two (or more) relations $\mathbf{S}_1$ to $\mathbf{S}_n$ specialize a single relation **T** (e.g., **PP** and **=** specialize **P**), then the disjunction of the specialized relations is equivalent to the target relation:

$$\mathbf{T}xy \leftrightarrow \mathbf{S}_1xy \vee \cdots \vee \mathbf{S}_nxy$$

For instance, $\mathbf{P}xy \leftrightarrow \mathbf{PP}xy \vee x = y$.

If exactly one relation holds between any two spatial regions, they are jointly exhaustive and pairwise disjoint (**JEPD**), captured by rules III (**JE**) and IV (**PD**), respectively. The former checks the satisfaction of at least one of the $\mathbf{R}_i$ relations in a domain for every pair of regions, while the latter checks that no spatial configuration satisfies more than one $\mathbf{R}_i$ at a time.

III) $\mathbf{R}_1xy \vee \mathbf{R}_2xy \vee ... \vee \mathbf{R}_nxy$ (jointly exhaustive),

IV) $\mathbf{R}_ixy \rightarrow \neg\mathbf{R}_jxy$ (pairwise disjoint).

By checking the rules mentioned above over the set of introduced mereotopological relations, we obtain a set of relations {**D**, $\mathbf{TPP}^{-1}$, $\mathbf{I}^{-1}$, **=**, **I**, **TPP**, **OV**, **X**, **PO**, and **EC**} that are provably JEPD and form the ten base relations that can be used to construct a composition table. The lattice formed by these relations is shown in Figure 3. The most general and the most specific cases are connected directly to $\top$ (tautology) and $\bot$ (contradiction) respectively, The relations that are directly above the contradiction ($\bot$) are our formalism's base relations. The proof of the properties shown in the lattice of the relations is mostly straightforward from the definitions.

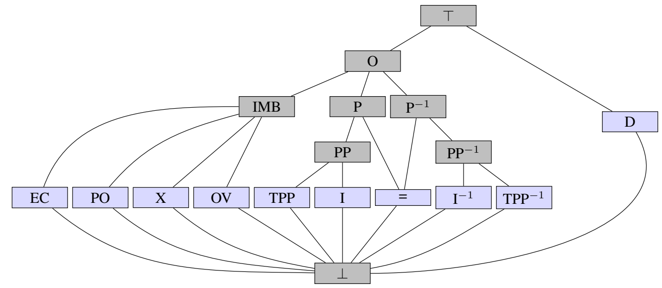

Figure 3: The lattice of the relations. The light colored boxes represent the JEPD set of relations.

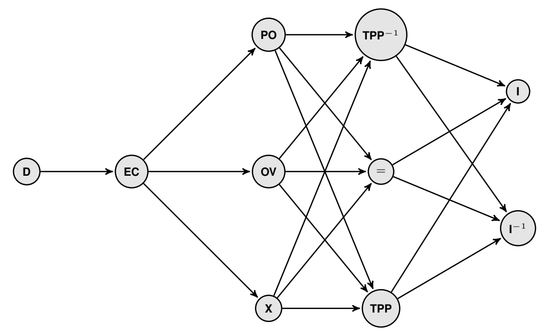

Figure 4: Conceptual network of topological relations.

The graph shown in Figure 4 depicts all possible topological transitions from the proposed relations in this theory. This graph is known as a continuity network and referred to by Freksa as a *conceptual neighborhood*. In this presentation, two JEPD relations are called conceptual neighbours if one can be converted into another through a gradual, ongoing change process that does not involve passing through any third relationship. Any subset of the JEPD relationships is a conceptual neighborhood where any member of the subset can form the two ends of a 'chain' where each pair of neighboring edges is conceptual neighbors. In terms of Figure 4, any linked sub-graph of the network forms a conceptual neighbor.

5 Logical Verification

To check the correctness of our formalism, we employ consistency checking, which is a standard technique for first-order logic formalisms. It confirms that the formalism does not entail any contradiction, that is, no formula (ϕ) exists such that both ϕ and $\neg\phi$ are logically entailed by the axioms and definitions.

This generally requires generating some small finite model via a finite model finder. We employ the Macleod suite of tools[6], which currently utilizes the finite model finders Mace4 [McCune, 2006] and Paradox3, and that has been previously used to prove the consistency of RCC-8 and some other even multidimensional theories.

We used the same approach to prove the consistency of our theory. However, the consistency checking approach is some-

[6]https://github.com/thahmann/macleod

what limited in its usefulness as it constructs only a single, and often the simplest, model. In such a model, many feature classes and spatial relations are empty or universal in every possible interpretation by using this technique. For example, one simple model for this formalism contains only a set of detached points, without any linear or areal features. The generated model also may not use most of the relations such as **X** or **EC** while some other relations such as **D** and **P** relate features only to themselves. Consequently, these relations could not be instantiated by all the feature classes of the domain. A set of more complex, non-trivial models can be generated by additional logical constraints. These restrictions ascertain that all relations can be instantiated both positively (i.e., a relation should hold be able to hold for some pair of regions) and negatively (i.e., a relation should not hold for all pairs of regions), by pairs of distinct objects from every feature classes. The creation of non-trivial models is forced by existential axioms of the form $\exists x\ P(x)$[7] and $\exists x \exists y\ [R(x, y) \wedge x \neq y]$[8]. This technique is also implemented in Macleod suite of tools and we used it to better ascertain the correctness of our set of relations.

6 Conclusions and Future Work

Mereotopology plays an important role in qualitative spatial representation. However, for a long time the theories were limited to equidimensional spatial entities. This study presents a mereotopological theory that is both formalized in first-order logic but that also yields a set of JEPD relations suitable for composition-based qualitative spatial reasoning over spatial entities of various dimensions. The theory is based on two primitives, "parthood" ($\mathbf{P}xy$) and "boundary part"($\mathbf{B}xy$). Based on these primitives, a set of ten JEPD relations are defined which are crucial in generating a composition table and lead to a lattice of relations. Furthermore, we constructed the conceptual neighbourhood graph of the relations that will be useful in constructing the composition table, which will be our immediate next step. Question raised by this study is how closely the introduced relations match those that are cognitively plausible for humans, which will be another line of future work. We also plan to study variations in the subjective understanding of human subjects.

Finally, this work could also be extended in the representation area by allowing objects that consist of a collection of entities (whether of the same or different dimensions) instead of only an isolated entity.

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[7] P denotes an arbitrary unary predicate, thus instantiating all classes.

[8] R denotes an arbitrary binary relation from the formalism.

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